Introduction to Mathematical Finance
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Practice questions 2020–2021

Proportional transaction costs

We consider the binomial financial market with a single period and two dates $t = 0$ and $t = 1$. The notations are the one used in the lectures.

However, unlike in the lectures, we suppose that for any transaction, the investor has to pay a fixed fee $c > 0$ per unit of risky asset S exchanged. In other words, when the investor buys Δ units of the risky asset S, he/she has to pay the transaction cost $c|\Delta|$, in addition to the cost of the assets.

As in the lectures, we denote by $X_1^{x,\Delta}$ the liquidative value of the portfolio with strategy (x, Δ) at time $t = 1$, *i.e.* the value that the investor gets at time $t = 1$ after **selling** (resp. **buying**) all the Δ shares of S that he/she holds (resp. has sold).

We aim at proving that there is no arbitrage opportunity if and only if

$$R + \frac{c(1+R)}{S_0} > d,\ R - \frac{c(1+R)}{S_0} < u. \tag{0.1}$$

1) Prove that

$$X_1^{x,\Delta} = \Delta S_1 + (x - \Delta S_0 - c|\Delta|)R - c|\Delta|,$$

for any strategy (x, Δ).

2) Suppose that (0.1) is satisfied.

   (a) Suppose that $\Delta \geq 0$. Prove that $\mathbb{P}[X_1^{0,\Delta} \geq 0] = 1$ implies $\Delta = 0$.

   (b) Suppose that $\Delta \leq 0$. Prove that $\mathbb{P}[X_1^{0,\Delta} \geq 0] = 1$ implies $\Delta = 0$.

   (c) Deduce that there is no arbitrage opportunity under condition (0.1).

3) We now consider the reverse implication.

   (a) Suppose that $R + \frac{c(1+R)}{S_0} \leq d$. Show that any $\Delta > 0$ defines an arbitrage opportunity.

   (b) Suppose that $R - \frac{c(1+R)}{S_0} \geq u$. Show that any $\Delta < 0$ defines an arbitrage opportunity.

   (c) Conclude.

4) Is the condition (0.1) different from that of the classical binomial model? What does condition (0.1) become in the case $c = 0$?

Quadratic risk minimisation and associated price

We consider the one-period trinomial model seen in the lectures. We recall that the corresponding market consists of a non-risky asset with price S^0 satisfying

$$S_0^0 = 1,\ S_1^0 = R,$$

and a risky asset whose price S_0 at time 0 is fixed and such that

$$\mathbb{P}(\{S_1 = uS_0\}) = p_1,\ \mathbb{P}(\{S_1 = dS_0\}) = p_2,\ \mathbb{P}(\{S_1 = S_0\}) = 1 - p_1 - p_2,$$

for some $0 < d < 1 < u$ and some $(p_1, p_2) \in (0,1)^2$, with $p_1 + p_2 < 1$.

1) Recall what is the no-arbitrage condition in this market.

2) We consider an option with maturity 1 and payoff $h(S_1)$. For a given initial capital $x \in \mathbb{R}$, we consider the following minimisation problem.

$$q^h(x) := \inf_{\Delta \in \mathbb{R}} \mathbb{E}^{\mathbb{P}} \left[\left(X_1^{x,\Delta} - h(S_1) \right)^2 \right].$$

Prove that the infimum above is attained at

$$\Delta^\star(x) := -\frac{\mathbb{E}^{\mathbb{P}}\left[(S_1 - S_0 R)(xR - h(S_1))\right]}{\mathbb{E}^{\mathbb{P}}\left[(S_1 - RS_0)^2\right]}.$$

3) Deduce that

$$q^h(x) = \mathbb{E}^{\mathbb{P}}\left[(xR - h(S_1))^2\right] - \frac{\left(\mathbb{E}^{\mathbb{P}}\left[(S_1 - RS_0)(xR - h(S_1))\right]\right)^2}{\mathbb{E}^{\mathbb{P}}\left[(S_1 - RS_0)^2\right]}.$$

4) We now consider the problem

$$m_{\text{quad}}(h) := \inf_{x \in \mathbb{R}} q^h(x).$$

Prove that this minimum is uniquely attained at some point $x^\star$.

5) Prove that

$$x^\star = \frac{1}{R}\left[q_1 h(uS_0) + q_2 h(dS_0) + (1 - q_1 - q_2) h(S_0)\right],$$

where

$$q_1 := \frac{p_1}{\mathbb{V}\text{ar}^{\mathbb{P}}[S_1 - RS_0]} \left(\mathbb{E}^{\mathbb{P}}\left[(S_1 - RS_0)^2\right] - S_0(u - R)\mathbb{E}^{\mathbb{P}}[S_1 - RS_0] \right),$$
$$q_2 := \frac{p_2}{\mathbb{V}\text{ar}^{\mathbb{P}}[S_1 - RS_0]} \left(\mathbb{E}^{\mathbb{P}}\left[(S_1 - RS_0)^2\right] + S_0(R - d)\mathbb{E}^{\mathbb{P}}[S_1 - RS_0] \right).$$

6) Under which condition(s) can $x^\star$ be interpreted as the expectation of the discounted payoff $h(S_1)$ under a specific risk-neutral measure? Comment and interpret.

Exercise 3: Jade Lizard strategy

A Jade Lizard strategy consists in buying a Call with maturity T and strike K_3, selling another Call option on the same underlying asset with maturity T and strike K_2 and finally selling a Put option on the same underlying asset with maturity T and strike K_1, such that

$$K_1 < K_2 < K_3, \text{ and } K_3 - K_2 < K_1.$$

Compute and represent both the total gain and the payoff of this strategy. What is its purpose?